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ELASTIC STATE OF A PLATE WITH A CIRCULAR PLUG AND A RECTILINEAR THIN ELASTIC INCLUSION*

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There is considered the problem of the state of stress of an infinite elastic plane with a bonded circular plug and an arbitrarily located thin elastic inclusion under biaxial tension. Conditions of ideal mechanical contact are satisfied on the line separating the materials. By using the complex Kolosov–Muskhelishvili potentials, the problem is reduced to a system of integro-differential equations which is solved numerically by utilization of a mechanical quadrature method. A numerical analysis is given for the solution of the problem of the elastic equilibrium of a plane with a circular hole and an arbitrarily located thin inclusion.

1. Let us consider the elastic equilibrium of an isotropic infinite plate with a bonded circular plug of radius R and an arbitrarily oriented rectilinear thin elastic inclusion of length $2l$ and width $2h$. The center of the plug, the point O, is connected to the Cartesian coordinate system xOy while the point O_1 at the center of the inclusion is the origin of a local coordinate system $x_1O_1y_1$, where the axis x_1 coincides with the middle line of the inclusion and makes an angle α with the x axis (Fig.1). The plate is stretched at infinity by uniformly distributed external forces N_1 and N_2 in mutually perpendicular directions, where the force N_1 makes the angle β with the x axis. On the line separating the plug from the plate the conditions of ideal mechanical contact are satisfied.

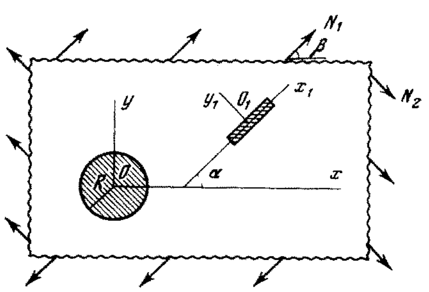

Fig.1.

We shall denote quantities characterizing the inclusion by the subscript 1, and the plug by 0. We use the superscripts plus and minus to denote the boundary values of the functions as $y_1 \to +0$ and $y_1 \to -0$, respectively. We denote the domain $|z| \leqslant R$ by S_0 and the domain $|z| \geqslant R$ by S_2. Here and henceforth, we retain the notation from the monograph /1/.

The following boundary conditions hold on the edges of the inclusion:

$$(\sigma_y - i\tau_{xy})^\pm = (\sigma_y - i\tau_{xy})_1^\pm, \quad (u + iv)^\pm = (u + iv)_1^\pm \tag{1.1}$$

The components $\sigma_x, \sigma_y, \tau_{xy}$ of the stress tensor and the components u, v of the displacement vector are defined from the formulas /1/

$$\begin{aligned} &\sigma_x + \sigma_y = 2\,[\Phi(z) + \overline{\Phi(z)}] \\ &\sigma_y - i\tau_{xy} = \Phi(z) + \Omega(\bar z) + (z - \bar z)\,\overline{\Phi'(z)} \\ &2\mu \frac{\partial}{\partial x}(u + iv) = \varkappa\Phi(z) - \Omega(\bar z) - (z - \bar z)\,\overline{\Phi'(z)} \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\Omega(z) = \overline{\Phi}(z) + z\overline{\Phi}'(z) + \overline{\Psi}(z) \\ &\Psi(z) = \overline{\Omega}(z) - \Phi(z) - z\Phi'(z) \end{aligned} \tag{1.3}$$

Because of the linearity of the problem, the complex potentials $\Phi(z)$ and $\Psi(z)$ are represented as follows for the plate

$$\Phi(z) = \Phi_1(z) + \Phi_2(z), \quad \Psi(z) = \Psi_1(z) + \Psi_2(z) \tag{1.4}$$

where $\Phi_j(z), \Psi_j(z)$ $(j = 1, 2)$ are functions governing the state of stress in a plate with an inclusion but without a circular plug $(j = 1)$ and with a circular plug but without the inclusion $(j = 2)$.

Neglecting higher order of smallness quantities compared with h for a thin inclusion, on

*Prikl.Matem.Mekhan.,Vol.46,No.6,pp.993-1000,1982 the basis of (1.2) it is possible to write in the $x_1O_1y_1$ coordinate system

$$(\sigma_y - i\tau_{xy})_1^+ - (\sigma_y - i\tau_{xy})_1^- = 2ihK'(x), \quad |x| \leqslant l \tag{1.5}$$
$$\frac{\partial}{\partial x}(u+iv)_1^+ - \frac{\partial}{\partial x}(u+iv)_1^- = \frac{ih}{\mu_1}M'(x), \quad |x| \leqslant l$$
$$(\sigma_y - i\tau_{xy})_1^+ + (\sigma_y - i\tau_{xy})_1^- = \frac{2}{(1+\varkappa_1)}[(1-\varkappa_1)K(x) + 2M(x) + 2\overline{K(x)} + 2\overline{M(x)}], \quad |x| \leqslant l$$
$$\frac{\partial}{\partial x}(u+iv)_1^+ + \frac{\partial}{\partial x}(u+iv)_1^- = \frac{1}{\mu_1(1+\varkappa_1)}[2\varkappa_1 K(x) + (\varkappa_1 - 1)M(x) - 2K(x) - 2M(x)] + i\frac{\gamma}{\mu_1}, \ |x| \leqslant l$$

Here $K(x)$, $M(x)$ are functions to be determined, and γ is the turning of the inclusion as a rigid whole. For simplicity, the subscript one is omitted from the variable x in (1.5) here and henceforth.

Using the boundary conditions (1.1), we obtain the following boundary value problem for the determination of the piecewise-holomorphic functions $\Phi_1(z)$, $\Omega_1(z)$ with the line of jumps $[-l, l]$ from the relations (1.5) and (1.2):

$$[\Phi_1(x) - \Omega_1(x)]^+ - [\Phi_1(x) - \Omega_1(x)]^- = 2ihK'(x), \ |x| \leqslant l \tag{1.6}$$
$$[\varkappa\Phi_1(x) + \Omega_1(x)]^+ - [\varkappa\Phi_1(x) + \Omega_1(x)]^- = 2ihkM'(x), \ |x| \leqslant l$$

$$[\Phi_1(x) + \Omega_1(x)]^+ + [\Phi_1(x) + \Omega_1(x)]^- = \frac{2}{1+\varkappa_1}[(1-\varkappa_1)K(x) + 2M(x) + 2\overline{K(x)} + 2\overline{M(x)}] - 2R(x)\varepsilon_1, \quad |x| \leqslant l; \quad \varepsilon_1 = 1 - \frac{\min(\mu, \mu_1)}{\mu} \tag{1.7}$$
$$[\varkappa\Phi_1(x) - \Omega_1(x)]^+ + [\varkappa\Phi_1(x) - \Omega_1(x)]^- = 2ik\gamma + \frac{2k}{1+\varkappa_1} \times [2\varkappa_1 K(x) + (\varkappa_1 - 1)M(x) - 2\overline{K(x)} - 2\overline{M(x)}] - 2P(x)\varepsilon_2, \quad |x| \leqslant l; \quad \varepsilon_2 = 1 - \frac{\min(\mu, \mu_1)}{\mu_1}$$

$$R(x) = \Phi_2(X) + \overline{\Phi_2(X)} + e^{-2i\alpha}(\overline{\Psi_2(X)} + X\overline{\Phi_2'(X)}) \tag{1.8}$$
$$P(x) = (1+\varkappa)\Phi_2(X) - R(x); \quad X = xe^{i\alpha} + z_0, \quad z_0 = x_0 + iy_0, \quad k = \mu/\mu_1$$

(x_0, y_0 are coordinates of the point O_1 in the xOy coordinate system).

Solving the linear conjugate problem (1.6) and going over to the xOy coordinate system, we obtain expressions for the functions $\Phi_1(z)$ and $\Psi_1(z)$:

$$\Phi_1(z) = \frac{h}{\pi(1+\varkappa)}\int_{-l}^{l}[K'(t) + kM'(t)]\frac{dt}{t - z_1} \tag{1.9}$$
$$\Psi_1(z) = \frac{h}{\pi(1+\varkappa)}\int_{-l}^{l}\left\{\frac{-\varkappa\overline{K'(t)} + k\overline{M'(t)}}{t - z_1} - \frac{\overline{T}e^{i\alpha}[K'(t) + kM'(t)]}{(t-z_1)^2}\right\}dt$$
$$T = te^{i\alpha} + z_0, \quad z_1 = e^{-i\alpha}(z - z_0)$$

We continue the function $\Phi_j(z)$ analytically from the domain S_j, into the domain S_{2-j} $(j = 0,2)$ by means of the formula

$$\Phi_j(z) = -\overline{\Phi}_j\left(\frac{R^2}{z}\right) + \frac{R^2}{z}\overline{\Phi}_j'\left(\frac{R^2}{z}\right) + \frac{R^2}{z^2}\overline{\Psi}_j\left(\frac{R^2}{z}\right) \tag{1.10}$$

and taking into account the relations (1.4), then for determination of tensor stress component σ_r and $\tau_{r\theta}$ in the polar system, we will have the following relations /2/:

$$\sigma_r + i\tau_{r\theta} = \left\{\Phi_j(z) - \frac{R^2}{r^2}\Phi_j\left(\frac{R^2}{\overline{z}}\right) + \left(1 - \frac{R^2}{r^2}\right)[\overline{\Phi_j(z)} - \overline{z}\overline{\Phi_j'(z)}]\right\} + \left[\Phi_1(z) + \overline{\Phi_1(z)} - \overline{z}\overline{\Phi_1'(z)} - \frac{\overline{z}}{z}\overline{\Psi_1(z)}\right]\delta_{j2}, \quad j = 0, 2 \tag{1.11}$$
$$2\mu_p\frac{\partial}{\partial\theta}(u+iv) = iz\left\{\left[\varkappa_p\Phi_j(z) + \frac{R^2}{r^2}\Phi_j\left(\frac{R^2}{\overline{z}}\right) - \left(1 - \frac{R^2}{r^2}\right)(\overline{\Phi_j(z)} - \overline{z}\overline{\Phi_j'(z)})\right] + \left[\varkappa_p\Phi_1(z) - \overline{\Phi_1(z)} + \overline{z}\overline{\Phi_1'(z)} + \frac{\overline{z}}{z}\overline{\Psi_1(z)}\right]\delta_{j2}\right\}$$
$$j = 0, 2$$
$$\delta_{ij} = \begin{cases}1, & i = j\\ 0, & i \neq j\end{cases}, \quad (\mu_p, \varkappa_p) = \begin{cases}(\mu, \varkappa), & j = 2\\ (\mu_0, \varkappa_0), & j = 0\end{cases}$$

Here $j = 2$ for the plane with the inclusion without the plug, $j = 0$ for the plug, $\Phi_0(z)$, $\Psi_0(z)$ are functions governing the state of stress in the plug. Ideal mechanical contact conditions are satisfied on the line separating the plug from the plate according to the condition of the problem, hence, on the basis of (1.11) we arrive at boundary value problems to determine the functions $\Phi_0(z)$ and $\Phi_2(z)$:

$$\begin{aligned}&[\Phi_0(t) + \Phi_2(t)]^+ - [\Phi_0(t) + \Phi_2(t)]^- = \\ &\quad \Phi_1(t) + \overline{\Phi_1(t)} - \overline{t\Phi_1'(t)} - \bar{t}t^{-1}\overline{\Psi_1(t)}, \quad t \in S_0 \cap S_2 \\ &[\mu\varkappa_0\Phi_0(t) - \mu_0\Phi_2(t)]^+ - [\mu_0\varkappa\Phi_2(t) - \mu\Phi_0(t)]^- = \\ &\quad \mu_0[\varkappa\Phi_1(t) - \overline{\Phi_1(t)} + \bar{t}\,\overline{\Phi_1'(t)} + \bar{t}t^{-1}\overline{\Psi_1(t)}], \quad t \in S_0 \cap S_2\end{aligned} \tag{1.12}$$

Solving the linear conjugate problems (1.12), and taking the relation (1.10) and the asymptotic representation of the functions $\Phi_j(z)$, $\Psi_j(z)$ $(j = 0, 2)$ into account /1/, we find after manipulations

$$\begin{aligned}&\Phi_2(z) = \Gamma + c\Big[\frac{R^2}{z^2}\overline{\Gamma}' + \overline{\Phi_1(0)} - \overline{\Phi}_1\Big(\frac{R^2}{z}\Big) + \\ &\quad \frac{R^2}{z}\overline{\Phi}_1'\Big(\frac{R^2}{z}\Big) + \frac{R^2}{z^2}\overline{\Psi}_1\Big(\frac{R^2}{z}\Big)\Big] \\ &\Psi_2(z) = L(z) + \frac{R^2}{z^2}\Big\{\Phi_2(z) + c_1\overline{\Phi}_1\Big(\frac{R^2}{z}\Big) + \frac{R^2}{z^2}\Big[R^2\overline{\Phi}_1''\Big(\frac{R^2}{z}\Big) + \\ &\quad 2\overline{\Psi}_1\frac{R^2}{z} + \frac{R^2}{z}\overline{\Psi}_1'\Big(\frac{R^2}{z}\Big)\Big]\Big\} \\ &L(z) = \Gamma' + \frac{R^2}{z^2}\Big[\Phi_1(0) + c_1\Gamma + c_2\overline{B}_0 + \frac{2R^2c}{z^2}\overline{\Gamma}'\Big] \\ &\Gamma = \frac{1}{4}(N_1 + N_2), \quad \Gamma' = \frac{1}{2}(N_1 - N_2)e^{-2i\beta}, \quad c = \frac{\mu - \mu_0}{\varkappa\mu_0 + \mu} \\ &B_0 = c_3\Big[\frac{(\mu_0 - \mu)\Phi_1(0) - (\varkappa_0\mu + \mu_0)\overline{\Phi_1(0)}}{\mu(1 + \varkappa_0)} - \Gamma\Big] \\ &c_1 = \frac{\mu\varkappa_0 - \mu_0\varkappa}{\varkappa_0\mu + \mu_0}, \quad c_2 = \frac{\mu(1 + \varkappa_0)}{\varkappa_0\mu + \mu_0}, \quad c_3 = \frac{\mu_0(1 + \varkappa)}{\mu\varkappa_0 + 2\mu_0 - \mu}\end{aligned} \tag{1.13}$$

Passing to the limit as $z_0 \to \infty$ in (1.13), we obtain values of the complex potentials for a plane with a bonded circular plug that agree with the corresponding formulas presented in /3/.

With the expressions (1.13) for $\Phi_2(z)$ and $\Psi_2(z)$ available, on the basis of (1.7) and (1.9) we obtain a system of integro-differential equations to determine the unknown functions $K(x)$, $M(x)$, which will have the following form in the dimensionless variables $\xi = x/l$, $\tau = t/l$

$$\sum_{j=1}^{4}\alpha_{ij}f_j(\xi) - \beta_i\int_{-1}^{1}\Big[\sum_{j=1}^{4}S_{ij}(\tau,\xi)f_j'(\tau)\Big]d\tau = p_i(\xi) \quad (i = 1, 2) \tag{1.14}$$

$$|\xi| \leqslant 1$$

$$\begin{aligned}&f_1(\xi) = M(l\xi), \quad f_2(\xi) = \overline{M(l\xi)}, \quad f_3(\xi) = K(l\xi), \quad f_4(\xi) = \overline{K(l\xi)} \\ &\beta_i = \frac{h\varepsilon_i}{l\pi(1 + \varkappa)}, \quad \alpha_{11} = \alpha_{12} = \alpha_{14} = \frac{2}{1 + \varkappa_1}, \quad \alpha_{13} = \frac{1 - \varkappa_1}{1 + \varkappa_1} \\ &\alpha_{22} = \alpha_{24} = -\frac{2k}{1 + \varkappa_1}, \quad \alpha_{21} = \frac{\varkappa_1 - 1}{1 + \varkappa_1}k, \quad \alpha_{23} = \frac{2\varkappa_1 k}{1 + \varkappa_1}\end{aligned} \tag{1.15}$$

$$\begin{aligned}&S_{1j}(\tau,\xi) = \Big[\frac{2}{\varepsilon_1}\frac{\delta_{1j}}{\tau - \xi} + G_j(\tau,\xi) + f_j(\tau,\xi) + g_j(\tau,\xi)\Big]k, \quad j = 1, 2 \\ &S_{13}(\tau,\xi) = \frac{1 - \varkappa}{\varepsilon_1}\frac{1}{\tau - \xi} + G_1(\tau,\xi) - \varkappa g_1(\tau,\xi) + f_1(\tau,\xi) \\ &S_{14}(\tau,\xi) = -\varkappa G_2(\tau,\xi) + g_2(\tau,\xi) + f_2(\tau,\xi) \\ &S_{2j}(\tau,\xi) = \Big[\frac{\nu - 1}{\varepsilon_2}\frac{\delta_{1j}}{\tau - \xi} - G_j(\tau,\xi) + \varkappa g_j(\tau,\xi) - \\ &\quad f_j(\tau,\xi) + f_3(\tau,\xi)\delta_{2j}\Big]k, \quad j = 1, 2 \\ &S_{23}(\tau,\xi) = \frac{2\varkappa}{\varepsilon_2}\frac{1}{\tau - \xi} - G_1(\tau,\xi) - \varkappa^2 g_1(\tau,\xi) - f_1(\tau,\xi) \\ &S_{24}(\tau,\xi) = \varkappa G_2(\tau,\xi) + \varkappa g_2(\tau,\xi) - f_2(\tau,\xi) + f_3(\tau,\xi)\end{aligned} \tag{1.16}$$

$$g_1(\tau,\xi)=\frac{c\varepsilon^2e^{i\alpha}}{X(X\bar{T}-\varepsilon^2)},\quad g_2(\tau,\xi)=ce^{-i\alpha}\left[\frac{\varepsilon^2(\varepsilon^2-T\bar{T})}{\bar{T}(\varepsilon^2-X\bar{T})^2}-\frac{1}{\bar{T}}\right] \tag{1.17}$$

$$G_1(\tau,\xi)=\frac{c_1}{c}\overline{g_1(\tau,\xi)}+\left(1+\frac{\varepsilon^2e^{-2i\alpha}}{\bar{X}^2}\right)\overline{g_2(\tau,\xi)}+g_3(\tau,\xi)$$

$$G_2(\tau,\xi)=\overline{g_1(\tau,\xi)}+g_4(\tau,\xi)$$

$$g_3(\tau,\xi)=ce^{-i\alpha}\frac{2\varepsilon^2(\varepsilon^2-X\bar{X})(\varepsilon^2-T\bar{T})}{\bar{X}(\varepsilon^2-T\bar{X})^3}$$

$$g_4(\tau,\xi)=ce^{-3i\alpha}\varepsilon^2\frac{X\bar{X}(\varepsilon^2-2T\bar{X})+3\varepsilon^2T\bar{X}-2\varepsilon^4}{\bar{X}^3(\varepsilon^2-T\bar{X})^3}$$

$$f_1(\tau,\xi)=\left[c+\frac{\varepsilon^2e^{-2i\alpha}}{\bar{X}^2}(c+c_4)\right]\frac{e^{i\alpha}}{T},\quad c_4=\frac{c_3(\mu_0-\mu)}{\varkappa_0\mu+\mu_0}$$

$$f_2(\tau,\xi)=\left[c+\frac{\varepsilon^2e^{-2i\alpha}}{\bar{X}^2}(1-c_3)\right]\frac{e^{-i\alpha}}{\bar{T}},\quad f_3(\tau,\xi)=(1+\varkappa)c\,\frac{e^{-i\alpha}}{\bar{T}}$$

$$p_i(\xi)=(-1)^{(i-1)}\varepsilon_i\Big\{2\Gamma+\bar{\Gamma}'e^{-2i\alpha}+c\varepsilon^2\Big[\frac{\bar{\Gamma}'}{\bar{X}^2}+\frac{\Gamma'}{X^2}+ \tag{1.18}$$
$$\frac{\Gamma'e^{-2i\alpha}}{\bar{X}^3}\left(\frac{3\varepsilon^2}{X}-2X\right)\Big]+\frac{\varepsilon^2e^{-2i\alpha}}{\bar{X}^2}(1-c_2c_3+c_1)\Gamma-$$
$$(1+\varkappa)\left(\Gamma+\frac{c\varepsilon^2\bar{\Gamma}'}{\bar{X}^2}\right)\delta_{2i}\Big\}-2ik\gamma\delta_{2i}$$

$$\varepsilon=R/l,\quad X=\xi e^{i\alpha}+z_0/l,\quad T=\tau e^{i\alpha}+z_0/l$$

The following conditions should hence be satisfied: uniqueness of the displacements when traversing the outline of the inclusion and equality to zero of the principal vector and principal moment of all the forces applied to the inclusion. These conditions can be represented in the form (Λ is a closed contour enclosing the domain of the inclusion)

$$\int_{-1}^{1}f_j'(\tau)\,d\tau=0,\quad j=1,3;\quad \operatorname{Re}\int_{\Lambda}[z_1\bar{\Omega}(z_1)+\Phi(z_1)]\,dz_1=0 \tag{1.19}$$

The system of equations (1.14) and the conditions (1.19) were solved numerically by using the method of mechanical quadratures /4/. After manipulation, we obtain a system of linear algebraic equations to determine u_{jm} and γ

$$\sum_{m=1}^{M}\sum_{j=1}^{4}M_{ij}(t_m,x_r)\,u_{jm}=Mp_i(x_r,\gamma);\quad i=1,2; \tag{1.20}$$
$$r=1,2,\ldots,\quad M-1$$
$$\sum_{m=1}^{M}u_{jm}=0,\quad j=1,3;\quad \operatorname{Im}\sum_{m=1}^{M}u_{3m}t_m=0$$
$$u_{jm}=f_j'(t_m)\sqrt{1-t_m^2},\quad t_m=\cos\frac{2m-1}{2M}\pi,\quad x_r=\cos\frac{\pi r}{M}$$
$$M_{ij}(t_m,x_r)=\alpha_{ij}\eta(t_m-x_r)-\beta_iS_{ij}(t_m,x_r)$$
$$\eta(t_m-x_r)=\begin{cases}0, & t_m>x_r\\ 1, & t_m\leqslant x_r\end{cases}$$

The state of stress in the neighborhood of the end of the inclusion can be represented by formulas in /5/, where the stress intensity factors K_i ($i=1,2,3,4$) are evaluated in the case under consideration by the formulas ($j=1$ for the left end, $j=2$ for the right end, M is even)

$$K_1^j-iK_2^j=k\Sigma_1,\quad K_3^j-iK_4^j=\Sigma_3$$
$$\Sigma_i=\frac{2h}{\sqrt{l}(1+\varkappa)}\frac{1}{M}\sum_{m=1}^{M}(-1)^m u_{im}\left(\operatorname{ctg}\frac{2m-1}{4M}\pi\right)^{(2j-3)},\quad i=1,3$$

The problem for an inclusion located within the plug is solved analogously. The solution of the problem when there are N inclusions in the plate of plug can be obtained by the same means.

2. Particular case. Plate with a plug and a crack. Introducing the change of variables

$$g_j'(\tau) = -f_j'(\tau)\, k \frac{2h}{1+\varkappa} \tag{2.1}$$

and passing to the limit as $\mu_1 \to 0$ in (1.14), we obtain

$$\int_{-1}^{1} [S_{11}(\tau,\xi)\, g_1'(\tau) + S_{12}(\tau,\xi)\, g_2'(\tau)]\, d\tau = \pi p_1(\xi) \tag{2.2}$$

$$g_3(\xi) = g_4(\xi) = 0, \quad |\xi| \leqslant 1$$

where the expressions for the functions $S_{11}(\tau,\xi), S_{12}(\tau,\xi), p_1(\xi)$ are given by (1.16)—(1.18).

Plate with a plug and an absolutely rigid inclusion. Passing to the limit as $\mu_1 \to \infty$ in (1.14), we find

$$\frac{h}{l} \int_{-1}^{1} [S_{23}(\tau,\xi)\, f_3'(\tau) + S_{24}(\tau,\xi)\, f_4'(\tau)]\, d\tau = \pi(1+\varkappa)\, p_2(\xi), \quad |\xi| \leqslant 1 \tag{2.3}$$

$$\sum_{j=1}^{4} \alpha_{1j} f_j(\xi) - \frac{h(1-\varkappa)}{l\pi(1+\varkappa)} \int_{-1}^{1} \frac{f_3'(\tau)}{\tau - \xi}\, d\tau = 0, \quad |\xi| \leqslant 1$$

where the expressions for the functions $\alpha_{1j}, S_{23}(\tau,\xi), S_{24}(\tau,\xi)$ are determined by (1.15)—(1.18).

Plate with a circular hole and an inclusion. Passing to the limit as $\mu_0 \to 0$ in (1.14), we will have a system of integro-differential equations for the plane with the circular hole and arbitrarily oriented inclusion under the assumption that the hole outline is force-free. We should set $c = c_1 = 1$ and

$$p_i(\xi) = (-1)^{(i-1)} \varepsilon_i \Big\{ 2\Gamma + \bar{\Gamma}' e^{-2i\alpha} + \varepsilon^2 \Big[\frac{\bar{\Gamma}'}{X^2} + \frac{\Gamma'}{\bar{X}^2} + \tag{2.4}$$

$$e^{-2i\alpha} \Big(\frac{\Gamma'}{\bar{X}^3} \Big(\frac{3\varepsilon^2}{\bar{X}} - 2X \Big) + \frac{2\Gamma}{\bar{X}^2} \Big) \Big] - (1+\varkappa)\, \delta_{2j} \Big(\Gamma + \frac{\varepsilon^2}{X^2} \bar{\Gamma}' \Big) \Big\} - 2ik\gamma\delta_{2i}$$

into (1.14)—(1.17) formulas.

Now passing to the limit as $\mu_1 \to 0$ we obtain an integral equation for a plane with a circular hole and an arbitrarily located crack, which agrees with the corresponding equation presented in /4/.

Passing to the limit as $\mu_1 \to \infty$, we find a system of integral equations for a plane with a circular hole and an absolutely rigid inclusion.

Plate with an inclusion. Passing to the limit as $\varepsilon \to 0$ in (1.14) or as $\mu_0 \to \mu$, we obtain a system of integro-differential equations of Prandtl type for a plane with an elastic inclusion, which agrees with the system of equations presented in /5/.

Two bonded half-planes with an inclusion. Performing the transformation of coordinate systems $x \to x,\ y \to y - \varepsilon$ and passing to the limit as $\varepsilon \to \infty$ in (1.14), we obtain a system of integro-differential equations for two homogeneous half-planes bonded along the real axis and with an arbitrarily located on elastic inclusion in one of them. In this case the relationships (1.17)—(1.18) have the form

$$g_1(\tau,\xi) = \frac{c e^{i\alpha}}{X - \bar{T}}, \quad g_2(\tau,\xi) = \frac{c e^{-i\alpha}(T - \bar{T})}{(X - \bar{T})^2} \tag{2.5}$$

$$G_1(\tau,\xi) = \frac{c_1}{c} \overline{g_1(\tau,\xi)} + (1 + e^{-2i\alpha}) \overline{g_2(\tau,\xi)} + g_3(\tau,\xi)$$

$$G_2(\tau,\xi) = \overline{g_1(\tau,\xi)} + g_4(\tau,\xi), \quad f_i(\tau,\xi) = 0 \quad (i = 1, 2, 3)$$

$$g_3(\tau,\xi) = \frac{2ce^{-i\alpha}(T - X)(\bar{T} - T)}{(\bar{X} - T)^3}, \quad g_4(\tau,\xi) = \frac{c(T - X)}{(\bar{X} - T)^2} e^{-3i\alpha}$$

$$p_i(\xi) = (-1)^{(i-1)}\varepsilon_i \{[2 + e^{-2i\alpha}(1 - c_2c_3 + c_1)]\Gamma + (1 - c)\bar{\Gamma}' e^{-2i\alpha} + \tag{2.6}$$

$$c(1 + e^{-2i\alpha})(\Gamma' + \bar{\Gamma}') - (1+\varkappa)(\Gamma + c\bar{\Gamma}')\delta_{2i}\} - 2ik\gamma\delta_{2i}$$

Half-plane with an elastic inclusion. Performing the transformation of coordinate systems as in the previous case, and passing to the limit as $\varepsilon \to \infty$ and $\mu_0 \to 0$ in (1.14)—(1.16), (2.5), (2.6) (i.e., setting $c = c_1 = 1$), we find a system of integro-differential equations for a half-plane with arbitrarily oriented elastic inclusion. In this case (2.6) takes the form

$$p_i(\xi) = (-1)^{(i-1)}\varepsilon_i \{[(1+e^{-2i\alpha})(2\Gamma + \Gamma' + \bar{\Gamma}')] - (1+\varkappa)(\Gamma + \bar{\Gamma}')\delta_{2i}\} - 2ik\gamma\delta_{2i} \qquad (2.7)$$

3. A numerical analysis of the solution of the problem has been performed. To 0.2% accuracy, values of the stress intensity factors were obtained for a crack and an absolutely rigid inclusion in an isotropic plane. Results of the numerical analysis for an elastic plane with a circular hole and arbitrarily oriented elastic inclusion are represented in Figs.2–4. Quantities referring to the left vertex of the inclusion are denoted by dashed, and the right by solid lines.

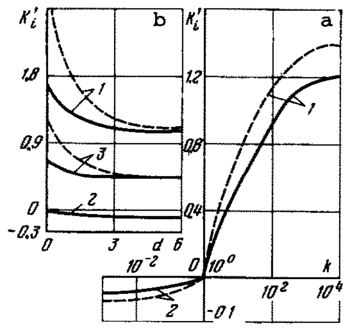

Fig.2

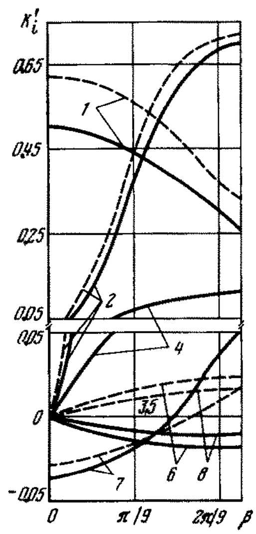

Fig.3

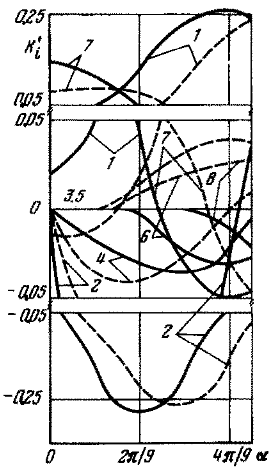

Fig.4

The calculations were performed for the following values of the parameters: $M = 20$, $h/l = 0.1$, $R/l = 2$, $x_0/l = 4$, $y_0/l = 0$, $N_2/N_1 = 0$, $\varkappa_1 = \varkappa = 2$.

The dependence of the stress intensity factors $K_i' = K_i / (\sqrt{l}N_1)$ $(i = 1, 2, 3, 4)$ on the relative plate stiffness $k = \mu/\mu_1$ is represented in Fig.2a for uniaxial tension in a direction perpendicular to the line of the inclusion $(\alpha = 0,\ \beta = \pi/2)$. Curves 1 and 2, respectively, characterize the stress intensity factors K_1' and K_2'. For such a loading $K_3' = K_4' = 0$.

The dependence of K_i' $(i = 1, 2, 3, 4)$ on the distance d between the edge of the hole and the left end of the inclusion is represented in Fig.2b under the condition that the inclusion is on the real axis for $\beta = \pi/2$. Curves 1 characterize the crack $(K_2' = K_3' = K_4' = 0)$, 2 is an absolutely stiff inclusion $(K_1' = K_2' = K_4' = 0)$, 3 is an elastic inclusion with the relative stiffness $k = 10$ $(K_2' = K_3' = K_4' = 0)$. For $k = 0.1$ the stress intensity factors are of the order of $10^{-4}-10^{-3}$ and, hence, are not indicated in Fig.2b.

Curves 1–4 in Figs.3 and 4 characterize the factors K_i' $(i = 1, 2, 3, 4)$ for $k = 10$, while curves 5–8 are the same factors for $k = 0.1$.

The dependence of K_i' on the angle β at which the tensile force acts for $\alpha = 0$ is given in Fig.3. Analyzing the shape of the functions $p_i(\xi)$ $(i = 1, 2)$, they can be represented as follows: $p_i(\xi) = A_i(\xi) + B_i(\xi)e^{2i\beta}$, where $A_i(\xi)$, $B_i(\xi)$ are certain real functions. We hence obtain

$$\operatorname{Re}\{p_i(\pi/4 - \beta)\} = A_i(\xi) + B_i(\xi)\sin 2\beta,\ \operatorname{Re}\{p_i(\pi/4 + \beta)\} = A_i(\xi) - B_i(\xi)\sin 2\beta,\ \ \operatorname{Im}\{p_i(\pi/4 - \beta)\} = \operatorname{Im}\{p_i(\pi/4 + \beta)\}$$

from which it follows that the straight line $\beta = \pi/4$ is the axis of symmetry for K_2', K_4', on the straight lines $K_i' = K_i'(\pi/4)$ $(i = 1,3)$ which are the axes of antisymmetry for K_1', K_3', respectively. Hence, it is sufficient to conduct investigations for the angles $0 \leqslant \beta \leqslant \pi/4$.

The dependence of K_i' $(i = 1, 2, 3, 4)$ on the angle of orientation α of the inclusion at $\beta = 0$ is represented in Fig.4. From physical considerations it follows that the mentioned dependences should be considered only in the segment $[0, \pi/2]$.

REFERENCES

1. MUSKHELISHVILI N.I., Certain Basic Problems of the Mathematical Theory of Elasticity, Groningen, Noordhoff, 1953.
2. PRUSOV I.A., Conguate Method in the Theory of Slabs. Beloruss.University Press, Minsk, 1975.

3. GRILITSKII D.V. and LUTSISHIN R.M., Stresses in Plates with a Circular Line of Separation of the Boundary Conditions. Vishcha Shkola, L'vov, 1975.
4. PANASIUK V.V., SAVRUK M.P. and DATSISHIN A.P., Stress Distribution Around Cracks in Plates and Shells, NAUKOVA DUMKA, Kiev, 1979.
5. DRAGAN M.S. and OPANASOVICH V.K., State of stress of a strip (beam) with rectilinear thin-walled inclusion, PMM, Vol.43, No.2, 1979, L'vov.

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